

# Reduction of Order, Periodicity and Boundedness in Nonlinear, Higher Order Difference Equations

H. SEDAGHAT <sup>1</sup>

## Abstract

We consider the semiconjugate factorization and reduction of order for non-autonomous, non-linear, higher order difference equations containing linear arguments. These equations have appeared in several mathematical models in biology and economics. By extending some recent results to cases where characteristic polynomials of the linear expressions have complex roots, we obtain new results on boundedness and the existence of periodic solutions for equations of order 3 or greater.

## 1 Introduction

Special cases of the following type of higher order difference equation have frequently appeared in the literature:

$$x_{n+1} = \sum_{i=0}^k a_i x_{n-i} + g_n \left( \sum_{i=0}^k b_i x_{n-i} \right), \quad n = 0, 1, 2, \dots \quad (1)$$

We assume here that  $k$  is a fixed positive integer and for each  $n$ , the function  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  is defined on the real line. The parameters  $a_i, b_i$  are fixed real numbers such that

$$a_k \neq 0 \text{ or } b_k \neq 0.$$

Upon iteration, Equation (1) generates a unique sequence of points  $\{x_n\}$  in  $\mathbb{R}$  (its solution) from any given set of  $k + 1$  initial values  $x_0, x_{-1}, \dots, x_{-k} \in \mathbb{R}$ . The number  $k + 1$  is the order of (1).

Special cases of Equation (1) appeared in the classical economic models of the business cycle in twentieth century in the works of Hicks [7], Puu [14], Samuelson [15] and others; see [18], Section 5.1 for some background and references. Other special cases of (1) occurred later in mathematical studies of biological models ranging from whale populations to neuron activity; see, e.g., Clark [1], Fisher and Goh [4], Hamaya [6] and Section 2.5 in Kocic and Ladas [10].

The dynamics of special cases of (1) have been investigated by several authors. Hamaya uses Liapunov and semicycle methods in [6] to obtain sufficient conditions for the global attractivity of

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<sup>1</sup>Department of Mathematics, Virginia Commonwealth University, Richmond, Virginia, 23284-2014, USA; Email: hsedagha@vcu.edu

the origin for the following special case of (1)

$$x_{n+1} = \alpha x_n + a \tanh \left( x_n - \sum_{i=1}^k b_i x_{n-i} \right)$$

with  $0 \leq \alpha < 1$ ,  $a > 0$  and  $b_i \geq 0$ . These results can also be obtained using only the contraction method in [13] and [17]; also see [12] for a discussion of alternative methods. The results in [17] are used in [18], Section 4.3D, to prove the global asymptotic stability of the origin for an autonomous special case of (1) with  $a_i, b_i \geq 0$  for all  $i$  and  $g_n = g$  for all  $n$ , where  $g$  is a continuous, non-negative function. The study of global attractivity and stability of fixed points for other special cases of (1) appear in [5] and [8]; also see [10], Section 6.9.

The second-order case ( $k = 1$ ) has been studied in greater depth. Kent and Sedaghat obtain sufficient conditions in [9] for the boundedness and global asymptotic stability of

$$x_{n+1} = cx_n + g(x_n - x_{n-1}) \quad (2)$$

In [3], El-Morshedy improves the convergence results of [9] for (2) and also gives necessary and sufficient conditions for the occurrence of oscillations. The boundedness of solutions of (2) is studied in [16] and periodic and monotone solutions of (2) are discussed in [19]. Li and Zhang study the bifurcations of solutions of (2) in [11]; their results include the Neimark-Sacker bifurcation (discrete analog of Hopf).

A more general form of (2), i.e., the following equation

$$x_{n+1} = ax_n + bx_{n-1} + g_n(x_n - cx_{n-1}) \quad (3)$$

is studied in [21] where sufficient conditions for the occurrence of periodic solutions, limit cycles and chaotic behavior are obtained using reduction of order and factorization of the above difference equation into a pair of equations of lower order. These methods are used in [2] to determine sufficient conditions on parameters for occurrence of limit cycles and chaos in those rational difference equations of the following type

$$x_{n+1} = \frac{ax_n^2 + bx_{n-1}^2 + cx_n x_{n-1} + dx_n + ex_{n-1} + f}{\alpha x_n + \beta x_{n-1} + \gamma} \quad (4)$$

that can be reduced to special cases of (3).

In this paper, we consider the possible occurrence of complex roots for the characteristic polynomials associated with the linear expressions  $\sum_{i=0}^k a_i u_i$  and  $\sum_{i=0}^k b_i u_i$  in (1). Complex, non-real roots that are common to both polynomials may occur when (1) has order 3 or greater ( $k \geq 2$ ), a situation that cannot occur in the second-order equations of [2] or [21]. We show that the resulting decomposition of (1) into lower-order equations via complex conjugate roots nevertheless yields a factor-cofactor pair in the sense of [20] or [22] with real coefficients and a type- $(k-1, 2)$  reduction of (1) within the real number system. We use this factorization to study boundedness and the occurrence of periodic solutions and limit cycles for (1). The results obtained here extend similar results in [21] to equations of order 3 and greater for the first time.

## 2 Reduction of order

We begin with a result from [22] (Theorem 5.6) that we quote here as a lemma. A generalization of this lemma to algebras over fields is proved in essentially the same way; see [23].

**Lemma 1** *Let  $g_n : \mathcal{F} \rightarrow \mathcal{F}$  be a sequence of functions on a field  $\mathcal{F}$ . If for  $a_i, b_i \in \mathcal{F}$  the polynomials*

$$P(u) = u^{k+1} - \sum_{i=0}^k a_i u^{k-i}, \quad Q(u) = \sum_{i=0}^k b_i u^{k-i}$$

*have a common, nonzero root  $\rho \in \mathcal{F}$  then each solution  $\{x_n\}$  of (1) in  $\mathcal{F}$  satisfies*

$$x_{n+1} = \rho x_n + t_{n+1} \tag{5}$$

*where the sequence  $\{t_n\}$  is the unique solution of the equation:*

$$t_{n+1} = - \sum_{i=0}^{k-1} p_i t_{n-i} + g_n \left( \sum_{i=0}^{k-1} q_i t_{n-i} \right) \tag{6}$$

*in  $\mathcal{F}$  with initial values  $t_{-i} = x_{-i} - \rho x_{-i-1}$  for  $i = 0, 1, \dots, k-1$  and coefficients*

$$p_i = \rho^{i+1} - a_0 \rho^i - \dots - a_i \quad \text{and} \quad q_i = b_0 \rho^i + b_1 \rho^{i-1} + \dots + b_i$$

*in  $\mathcal{F}$ . Conversely, if  $\{t_n\}$  is a solution of (6) with initial values  $t_{-i} \in \mathcal{F}$  then the sequence  $\{x_n\}$  that it generates in  $\mathcal{F}$  via (5) is a solution of (1).*

The preceding result shows that Equation (1) splits into the equivalent pair of equations (5) and (6) provided that the polynomials  $P$  and  $Q$  have a common nonzero root  $\rho$ . We call the pair of equations (5) and (6) a *semiconjugate factorization* of (1). Equation (6), whose order is one less than the order of (1) is the *factor equation* and Equation (5) which bridges the order (or dimension) gap between (1) and (6) is the *cofactor equation*.

Since Equation (6) is of the same type as (1) we may consider applying Lemma 1 to (6) to obtain a further reduction of order. This is done next.

**Lemma 2** *Let  $k \geq 2$  and assume that the coefficients of (1) are complex, i.e.,  $a_i, b_i \in \mathbb{C}$ . Let  $\mathcal{F} = \mathbb{C}$  in Lemma 1 and suppose that  $g_n : \mathbb{C} \rightarrow \mathbb{C}$  are complex functions for all  $n$ . If the polynomials  $P, Q$  in Lemma 1 have two common, nonzero roots  $\rho, \gamma \in \mathbb{C}$  then (6) has a factor equation*

$$r_{n+1} = - \sum_{j=0}^{k-2} p'_j r_{n-j} + g_n \left( \sum_{j=0}^{k-2} q'_j r_{n-j} \right) \tag{7}$$

with coefficients

$$p'_j = \gamma^{j+1} + p_0\gamma^j + \cdots + p_j \quad \text{and} \quad q'_j = q_0\gamma^j + q_1\gamma^{j-1} + \cdots + q_j.$$

where the numbers  $p_j, q_j$  are as defined in Lemma 1 in terms of the root  $\rho$ . There are two cofactor equations

$$t_{n+1} = \gamma t_n + r_{n+1} \tag{8}$$

$$x_{n+1} = \rho x_n + t_{n+1} \tag{9}$$

the second of which is just (5) from Lemma 1. The triangular system of three equations (7)-(9) is equivalent to (1) in the sense of Lemma 1; i.e., they generate the same set of solutions.

**Proof.** Consider the polynomials associated with the factor equation (6), i.e.,

$$P_1(u) = u^k + \sum_{j=0}^{k-1} p_j u^{k-j-1}, \quad Q_1(u) = \sum_{j=0}^{k-1} q_j u^{k-j-1}.$$

Let  $\rho$  be a root of  $P$ . We claim that

$$(u - \rho)P_1(u) = P(u).$$

This is established by straightforward calculation:

$$\begin{aligned} (u - \rho)P_1(u) &= (u - \rho) \left( u^k + \sum_{j=0}^{k-1} p_j u^{k-j-1} \right) \\ &= u^{k+1} - \rho p_{k-1} + \sum_{j=0}^{k-1} (p_j - \rho p_{j-1}) u^{k-j} \end{aligned}$$

where we define  $p_{-1} = 1$  to simplify the notation. Using the definition of the numbers  $p_i$  in Lemma 1 we obtain

$$p_j - \rho p_{j-1} = -a_j$$

and further, since  $P(\rho) = 0$  we obtain

$$\begin{aligned} \rho p_{k-1} &= \rho(\rho^k - a_0\rho^{k-1} - \cdots - a_{k-1}) \\ &= P(\rho) + a_k \\ &= a_k \end{aligned}$$

which completes the proof of the claim. A similar argument shows that if  $\rho$  is a root of  $Q$  then

$$(u - \rho)Q_1(u) = Q(u).$$

Now, suppose that  $\gamma$  is also a common root of  $P$  and  $Q$ . If  $\gamma \neq \rho$  then clearly  $P_1(\gamma) = Q_1(\gamma) = 0$  so  $\gamma$  is a common root of  $P_1$  and  $Q_1$ . Otherwise,  $\gamma = \rho$  and  $\rho$  is a double root, hence a zero of the derivatives  $P'$  and  $Q'$ , i.e.,

$$P'(\rho) = Q'(\rho) = 0.$$

In addition, we find that

$$\begin{aligned} P_1(\rho) &= \rho^k + \sum_{j=0}^{k-1} (\rho^{j+1} - a_0\rho^j - \dots - a_{j-1}\rho - a_j)\rho^{k-j-1} \\ &= (k+1)\rho^k - \sum_{j=0}^{k-1} (k-j)a_j\rho^{k-j-1} \\ &= P'(\rho) \end{aligned}$$

so that  $\rho = \gamma$  is a root of  $P_1$ . Similarly,  $\rho = \gamma$  is also seen to be a root of  $Q_1$ . Now applying Lemma 1 to (6) yields a factor equation (7) and a cofactor (8).

Finally, the last assertion follows from Theorem 3.1 in [22] (or Theorem 6 in [20]). ■

The next result on factorization of polynomials is also needed.

**Lemma 3** *Suppose that  $\gamma, \rho \in \mathbb{C}$  are roots of the polynomial  $c_0u^m + c_1u^{m-1} + \dots + c_{m-1}u + c_m$  of degree  $m \geq 2$  with coefficients  $c_j \in \mathbb{R}$ . Then*

$$\sum_{j=0}^m c_j u^{m-j} = (u^2 - (\gamma + \rho)u + \gamma\rho) \sum_{j=0}^{m-2} \alpha_j u^{m-j-2} \quad (10)$$

where  $\alpha_0 = c_0$ ,

$$\alpha_j = c_j + (\gamma + \rho)\alpha_{j-1} - \gamma\rho\alpha_{j-2}, \quad j = 1, 2, \dots, m-2, \quad \alpha_{-1} \doteq 0, \quad (11)$$

and the following equalities hold:

$$c_{m-1} + (\gamma + \rho)\alpha_{m-2} - \gamma\rho\alpha_{m-3} = 0, \quad (12)$$

$$c_m - \gamma\rho\alpha_{m-2} = 0. \quad (13)$$

Further, if  $\gamma$  and  $\rho$  are either both real or they are complex conjugates then the numbers  $\alpha_j$ ,  $j = 1, 2, \dots, m-2$  that satisfy the recursions (11) are real and found to be:

$$\alpha_j = \sum_{i=0}^j \frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} c_{j-i}, \quad \text{if } \gamma \neq \rho \quad (14)$$

$$\alpha_j = \sum_{i=0}^j (i+1)\rho^i c_{j-i}, \quad \text{if } \gamma = \rho. \quad (15)$$

Conversely, let  $\sum_{j=0}^m c_j u^{m-j}$  be a polynomial with real coefficients  $c_j$ . If  $\gamma, \rho \in \mathbb{C}$  and there are real numbers  $\alpha_j$  satisfying (11)-(13) then (10) holds and  $\gamma, \rho$  are roots of  $\sum_{j=0}^m c_j u^{m-j}$ .

**Proof.** Assume that  $\gamma, \rho \in \mathbb{C}$  are roots of  $\sum_{j=0}^m c_j u^{m-j}$ . Then this polynomial is evenly divided by the quadratic polynomial

$$(u - \gamma)(u - \rho) = u^2 - (\gamma + \rho)u + \gamma\rho \quad (16)$$

with a resulting quotient polynomial  $\sum_{j=0}^{m-2} \alpha_j u^{m-j-2}$ ; i.e., (10) holds. To determine the coefficients  $\alpha_j$  of the quotient, multiply the polynomials on the right hand side of (10) and rearrange terms to obtain the identity

$$\begin{aligned} \sum_{j=0}^m c_j u^{m-j} &= \alpha_0 u^m + (\alpha_1 - (\gamma + \rho)\alpha_0)u^{m-1} + (\alpha_2 - (\gamma + \rho)\alpha_1 + \gamma\rho\alpha_0)u^{m-2} \\ &\quad + \cdots + (\alpha_{m-2} - (\gamma + \rho)\alpha_{m-3} + \gamma\rho\alpha_{m-4})u^2 + \\ &\quad + (-(\gamma + \rho)\alpha_{m-2} + \gamma\rho\alpha_{m-3})u + \gamma\rho\alpha_{m-2}. \end{aligned}$$

Now, matching coefficients on the two sides yields (11)-(13).

Next, if  $\gamma$  and  $\rho$  are either both real or they are complex conjugates then  $(\gamma + \rho)$  and  $\gamma\rho$  are both real. In this case, the numbers  $\alpha_j$  defined by the recursions (11) are also real. Finally, (14) and (15) may be proved by induction. First, suppose that  $\gamma \neq \rho$ . For  $j = 1$  we have

$$c_1 + \frac{\gamma^2 - \rho^2}{\gamma - \rho}c_0 = c_1 + (\gamma + \rho)\alpha_0 = \alpha_1$$

so (14) is true if  $j = 1$ . Suppose next that for  $1 \leq j \leq m - 3$ , (14) is true for  $1, 2, \dots, j$ . Then for  $j + 1$

$$\begin{aligned} \alpha_{j+1} &= c_{j+1} + (\gamma + \rho)\alpha_j - \gamma\rho\alpha_{j-1} \\ &= c_{j+1} + (\gamma + \rho) \sum_{i=0}^j \frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} c_{j-i} - \gamma\rho \sum_{i=1}^j \frac{\gamma^i - \rho^i}{\gamma - \rho} c_{j-i} \\ &= c_{j+1} + (\gamma + \rho)c_j + \sum_{i=1}^j \left[ (\gamma + \rho) \frac{\gamma^{i+1} - \rho^{i+1}}{\gamma - \rho} - \gamma\rho \frac{\gamma^i - \rho^i}{\gamma - \rho} \right] c_{j-i}. \end{aligned}$$

Since for each  $i = 1, \dots, j$

$$(\gamma + \rho)(\gamma^{i+1} - \rho^{i+1}) - \gamma\rho(\gamma^i - \rho^i) = \gamma^{i+2} - \rho^{i+2}$$

we obtain

$$\alpha_{j+1} = c_{j+1} + \frac{\gamma^2 - \rho^2}{\gamma - \rho}c_j + \sum_{i=1}^j \frac{\gamma^{i+2} - \rho^{i+2}}{\gamma - \rho} c_{j-i}$$

which verifies the induction step. If  $\gamma = \rho$  then for  $j = 1$

$$c_1 + 2\rho c_0 = c_1 + 2\rho\alpha_0 = \alpha_1$$

so (15) is true if  $j = 1$ . Suppose next that for  $1 \leq j \leq m - 3$ , (15) is true for  $1, 2, \dots, j$ . Then for  $j + 1$

$$\begin{aligned}\alpha_{j+1} &= c_{j+1} + 2\rho\alpha_j - \rho^2\alpha_{j-1} \\ &= c_{j+1} + 2\rho \sum_{i=0}^j (i+1)\rho^i c_{j-i} - \rho^2 \sum_{i=1}^j i\rho^{i-1} c_{j-i} \\ &= c_{j+1} + 2\rho c_j + \sum_{i=1}^j [2(i+1)\rho^{i+1} - i\rho^{i+1}] c_{j-i} \\ &= c_{j+1} + 2\rho c_j + \sum_{i=1}^j (i+2)\rho^{i+1} c_{j-i}\end{aligned}$$

which verifies the induction step.

Conversely, if  $\gamma, \rho \in \mathbb{C}$  and  $\alpha_j \in \mathbb{R}$  satisfy (11)-(13) then by the definition of  $\alpha_j$  the quadratic polynomial (16) divides  $\sum_{j=0}^m c_j u^{m-j}$  evenly. Therefore,  $\gamma, \rho$  are roots of  $\sum_{j=0}^m c_j u^{m-j}$ . ■

If the coefficients  $a_i, b_i$  in (1) are real and a common root  $\rho$  of  $P$  and  $Q$  is complex then these polynomials also share another complex root, namely, the conjugate  $\bar{\rho}$ ; thus, Lemma 2 is applicable. However, if the functions  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  are real functions then a direct application of Lemma 2 is problematic since the coefficients  $p_i, q_i$  of the factor equation (6) are complex. The next result shows that this difficulty does not actually arise since the coefficients  $p'_i, q'_i$  of the secondary factor equation (7) are in fact, real and furthermore, the two complex cofactor equations in Lemma 2 combine into a single second-order cofactor equation in  $\mathbb{R}$ .

**Theorem 4** *Let  $k \geq 2$  in (1) and assume that the coefficients  $a_i, b_i$  are all real and  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  for  $n \geq 0$ . If the polynomials  $P, Q$  in Lemma 1 have a common complex root  $\rho = \mu e^{i\theta} \notin \mathbb{R}$  then the following statements are true:*

(a) *The coefficients  $p'_j, q'_j$  of the factor equation (7) in Lemma 2 are real numbers that may be writtern in terms of the original coefficients  $a_i, b_i$  of (1) as*

$$p'_j = \mu^{j+1} \frac{\sin(j+2)\theta}{\sin\theta} - \frac{1}{\sin\theta} \sum_{m=0}^j a_m \mu^{j-m} \sin(j-m+1)\theta, \quad (17)$$

$$q'_j = \frac{1}{\sin\theta} \sum_{m=0}^j b_m \mu^{j-m} \sin(j-m+1)\theta \quad (18)$$

for  $j = 0, 1, \dots, k-2$ .

(b) The pair of first-order cofactor equations in Lemma 2 with complex coefficients  $\rho$  and  $\gamma = \bar{\rho}$  combine into one equivalent, second-order, non-homogeneous linear equation with real coefficients

$$x_{n+1} - 2\mu \cos \theta x_n + \mu^2 x_{n-1} = r_{n+1} \quad (19)$$

where the sequence  $\{r_n\}$  is a solution of the factor equation (7) in  $\mathbb{R}$ .

(c) The system of equations (7) and (19) is equivalent to (1); i.e., the set of solutions of (19) with  $\{r_n\}$  satisfying (7) is identical with the set of solutions of (1).

**Proof.** (a) Let  $\rho = \mu e^{i\theta} = \mu \cos \theta + i\mu \sin \theta$  and  $\gamma = \bar{\rho}$  be complex conjugate roots of both  $P$  and  $Q$  with  $\sin \theta \neq 0$  since  $\rho \notin \mathbb{R}$ . Recall from the proof of Lemma 2 that

$$P(u) = (u - \rho)P_1(u), \quad Q(u) = (u - \rho)Q_1(u)$$

Applying the same argument to the polynomials  $P_1$  and  $Q_1$  using their common root  $\bar{\rho}$  yields

$$\begin{aligned} P(u) &= (u - \rho)(u - \bar{\rho})P_2(u) = (u^2 - (\rho + \bar{\rho})u + \rho\bar{\rho})P_2(u), \\ Q(u) &= (u - \rho)(u - \bar{\rho})Q_2(u) = (u^2 - (\rho + \bar{\rho})u + \rho\bar{\rho})Q_2(u) \end{aligned}$$

where

$$P_2(u) = u^{k-1} - \sum_{j=0}^{k-2} p'_j u^{k-j-2}, \quad Q_2(u) = \sum_{j=0}^{k-2} q'_j u^{k-j-2}.$$

Applying Lemma 3 to each of  $P$  and  $Q$  we obtain (17) and (18) from (14) since for every positive integer  $m$ ,

$$\frac{\gamma^m - \rho^m}{\gamma - \rho} = \frac{-\mu^m(e^{i\theta m} - e^{-i\theta m})}{-\mu(e^{i\theta} - e^{-i\theta})} = \mu^{m-1} \frac{\sin m\theta}{\sin \theta}.$$

(b) Eliminate  $t_{n+1}$  and  $t_n$  from (8) using (9) to obtain

$$\begin{aligned} x_{n+1} - \rho x_n &= \gamma(x_n - \rho x_{n-1}) + r_{n+1}, \quad \text{or:} \\ x_{n+1} - (\rho + \gamma)x_n + \rho\gamma x_{n-1} &= r_{n+1} \end{aligned}$$

which is the same as (19). Now if  $\{x_n\}$  is a solution of (19) with a given sequence  $\{r_n\}$  then by the preceding argument, the sequence  $\{x_n - \rho x_{n-1}\}$  satisfies (8). Further, with  $t_n = x_n - \rho x_{n-1}$  it is clear that  $\{x_n\}$  satisfies (9) so that the sequence of pairs  $\{(t_n, x_n)\}$  is a solution of the system of equations (8) and (9). Conversely, if  $\{(t_n, x_n)\}$  is a solution of the system then the above construction shows that  $\{x_n\}$  satisfies (19). Therefore, the same set of solutions  $\{x_n\}$  is obtained; i.e., the system is equivalent to the second-order equation.

(c) The equivalence of the system of equations (7) and (19) to (1) is a consequence of Theorem 3.1 in [22] (or Theorem 6 in [20]). ■



**Remarks.**

1. Theorem 4 shows that the existence of a common *complex* (non-real) root  $\rho$  for the polynomials  $P, Q$  leads to a type- $(k-1, 2)$  reduction, or factorization, of (1) over the real numbers. Over the field of complex numbers  $\mathbb{C}$ , this reduction is equivalent to repeated type- $(k, 1)$  reductions as outlined in Lemmas 1 and 2; see [22] for the general background on reduction types.
2. The parameters  $a_j, b_j$ ,  $j = k-1, k$  which affect  $\rho$  but do not appear in (17) and (18) are not free. They satisfy (12) and (13) in Lemma 3 and for the complex conjugate pair of roots in Theorem 4 they take the forms

$$a_k = p'_{k-2}\mu^2, \quad a_{k-1} = p'_{k-3}\mu^2 - 2p'_{k-2}\mu \cos \theta; \quad (20)$$

$$b_k = q'_{k-2}\mu^2, \quad b_{k-1} = q'_{k-3}\mu^2 - 2q'_{k-2}\mu \cos \theta. \quad (21)$$

Here we assume that  $p'_{-1} = 1$  and  $q'_{-1} = 0$  when  $k = 2$ .

### 3 Boundedness and periodicity

In this section we use reduction of order and factorization methods of the preceding section to prove the existence of oscillations in the real solutions of certain difference equations of type (1). Convergence and global attractivity issues regarding this equation are discussed in [23] in at a much more general level.

We quote the next result from the literature as a lemma; see [21] or Section 5.5 in [22]. This result pertains to Equation (9) whose solution may be written in the following way:

$$x_n = \rho^n x_0 + \sum_{j=1}^n \rho^{n-j} t_j. \quad (22)$$

**Lemma 5** (*periodicity, limit cycles, boundedness*) *Let  $p$  be a positive integer and let  $\rho \in \mathbb{C}$  with  $\rho \neq 0$ .*

(a) *If for a given sequence  $\{t_n\}$  of complex numbers Eq.(22) has a solution  $\{x_n\}$  of period  $p$  then  $\{t_n\}$  is periodic with period  $p$ .*

(b) *Let  $\{t_n\}$  be a periodic sequence of complex numbers with prime (or minimal) period  $p$  and assume that  $\rho$  is not a  $p$ -th root of unity; i.e.,  $\rho^p \neq 1$ . If  $\{\tau_0, \dots, \tau_{p-1}\}$  is one cycle of  $\{t_n\}$  and*

$$\xi_i = \frac{1}{1 - \rho^p} \sum_{j=0}^{p-1} \rho^{p-j-1} \tau_{(i+j) \bmod p} \quad i = 0, 1, \dots, p-1 \quad (23)$$

*then the solution  $\{x_n\}$  of Eq.(22) with  $x_0 = \xi_0$  and  $t_1 = \tau_0$  has prime period  $p$  and  $\{\xi_0, \dots, \xi_{p-1}\}$  is a cycle of  $\{x_n\}$ .*

(c) If  $|\rho| < 1$  and  $\{t_n\}$  is a sequence that converges to a  $p$ -cycle then the sequence  $\{x_n\}$  that is generated by (22) converges to a  $p$ -cycle. If  $\{\tau_0, \dots, \tau_{p-1}\}$  is one cycle of the limit of  $\{t_n\}$  then  $\{\xi_0, \dots, \xi_{p-1}\}$  is a cycle of the limit of  $\{x_n\}$  where  $\xi_i$  is defined by (23).

(d) If  $|\rho| < 1$  and  $\{t_n\}$  is a bounded sequence with  $|t_n| \leq M$  for all  $n$  then the sequence  $\{x_n\}$  that is generated by (22) is also bounded and there is a positive integer  $N$  such that

$$|x_n| \leq |\rho| + \frac{M}{1 - |\rho|} \quad \text{for all } n \geq N.$$

Lemma 5 and Theorem 4 imply the following result.

**Corollary 6** *Let  $k \geq 2$  in (1) and assume that the coefficients  $a_i, b_i$  are real and  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  for  $n \geq 0$ . If the polynomials  $P, Q$  in Lemma 1 have a common complex root  $\rho = \mu e^{i\theta} \notin \mathbb{R}$  then the following statements are true:*

(a) *If  $\rho$  is not a  $p$ -th root of unity then for every periodic solution of (7) of prime period  $p$  (1) has a periodic solution of prime period  $p$  that is given by (23).*

(b) *If modulus  $|\rho| < 1$  then for every limit cycle (attracting periodic solution) of (7) of period  $p$  (1) has a limit cycle of period  $p$ .*

(c) *If modulus  $|\rho| < 1$  then for every bounded solution of (7) the corresponding solution of (1) is bounded. Hence, if every solution of (7) is bounded then every solution of (1) is bounded.*

**Proof.** We prove only (a) since the proofs of (b) and (c) use similar reasoning using Lemma 5. Recall that the second-order cofactor equation (19) in Theorem 4 is equivalent to the pair of first-order cofactor equations (8) and (9). Let  $\{r_n\}$  be a solution of (7) having prime period  $p$ . If  $\rho$  is not a  $p$ -th root of unity then by Lemma 5(b) equation (8) has a solution  $\{t_n\}$  in  $\mathbb{C}$  with prime period  $p$ . Another application of Lemma 5 to equation (9) shows that the solution  $\{x_n\}$  of (19) in  $\mathbb{R}$  and hence, of (1) also has prime period  $p$ . ■

It is worth pointing out that if  $|\rho| \geq 1$  then the periodic solution of (1) in Corollary 6(a) is not attracting even if the corresponding solution  $\{r_n\}$  of (7) is attracting. Therefore, such solutions may be difficult to identify numerically. Only when  $|\rho| < 1$  and the homogeneous part of the cofactor equation (19) fades away do the solutions of the factor equation (7) determine the asymptotic behavior of solutions of (1).

In closing, we discuss the solutions of a third-order version of (1), i.e.,  $k = 2$  to illustrate the various aspects of the preceding results. Consider the autonomous difference equation

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + g(x_n + b_1 x_{n-1} + b_2 x_{n-2}) \quad (24)$$

where  $a_0, a_1, a_2, b_1, b_2 \in \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$ . If  $a_2 = b_2 = 0$  then (24) reduces to an autonomous version of the second-order equation (3). We assume here that  $b_2 \neq 0$ .

The polynomial  $Q$  of (24) is the quadratic  $u^2 + b_1u + b_2$  whose roots are complex if and only if  $b_1^2 < 4b_2$ . These complex conjugate roots are shared by the polynomial  $P$  if and only if conditions (20) hold. Since  $k = 2$  we calculate

$$\begin{aligned} p_0 &= \rho - a_0, & p_1 &= \rho^2 - a_0\rho - a_1 \\ q_0 &= 1, & q_1 &= \rho + b_1 \\ p'_0 &= \bar{\rho} + p_0 = -b_1 - a_0, & q'_0 &= 1 \end{aligned}$$

Note that  $\rho + \bar{\rho} = -b_1$  and  $\rho\bar{\rho} = b_2$ ,  
Thus conditions (20) in this case are

$$a_1 = b_1(a_0 + b_1) - b_2, \quad a_2 = b_2(a_0 + b_1). \quad (25)$$

We may alternatively obtain (25) using (12) and (13). If  $b_1^2 < 4b_2$  and conditions (25) hold then (24) is equivalent to the pair of equations

$$r_{n+1} = (a_0 + b_1)r_n + g(r_n), \quad (26)$$

$$x_{n+1} = -b_1 x_n - b_2 x_{n-1} + r_{n+1} \quad (27)$$

for  $n \geq 0$  where the initial value of (26) is  $r_0 = x_0 + b_1 x_{-1} + b_2 x_{-2}$  for a given triple of real initial values  $x_0, x_{-1}, x_{-2}$  for (24).

Next, suppose that  $g$  is a rational function of the following type

$$g(u) = \frac{A}{u} + B + Cu, \quad A, B, C \in \mathbb{R}, \quad A \neq 0. \quad (28)$$

With this  $g$  the difference equation is an example of a third-order rational recursive equation. A second-order version of this equation is a rational equation of type (4) that is studied in [2].

If  $B = 0$  and  $C = -a_0 - b_1$  then the factor equation (26) reduces to

$$r_{n+1} = \frac{A}{r_n}. \quad (29)$$

Every solution of (29) has period 2 with cycles  $\{r_0, A/r_0\}$  as long as  $r_0 \neq 0$ . Since  $\rho \neq \pm 1$ , corresponding to each solution of (29) with period two, the solution of (24) whose triple of initial values  $(x_{-2}, x_{-1}, x_0)$  is not on the plane  $u + b_1v + b_2w = 0$  (so that  $r_0 \neq 0$ ) has period 2. This plane which passes through the origin is in fact the singularity (or forbidden) set of (24) in this case. The aforementioned periodic solutions of (24) have cycles  $\{\xi_0, \xi_1\}$  that we calculate in two stages using (23). First, for (8) with  $\gamma = \bar{\rho}$  we calculate the cycles  $\{\tau_0, \tau_1\}$  in  $\mathbb{C}$  as

$$\tau_0 = \frac{\bar{\rho}r_0 + A/r_0}{1 - \bar{\rho}^2}, \quad \tau_1 = \frac{r_0 + \bar{\rho}A/r_0}{1 - \bar{\rho}^2}.$$

Next, using  $\{\tau_0, \tau_1\}$  in (23) we calculate the cycles  $\{\xi_0, \xi_1\}$  for (24)

$$\xi_0 = \frac{\rho\tau_0 + \tau_1}{1 - \rho^2} = \frac{\rho\bar{\rho}r_0 + \rho A/r_0 + r_0 + \bar{\rho}A/r_0}{(1 - \rho^2)(1 - \bar{\rho}^2)} = \frac{\rho\bar{\rho}r_0 + (\rho + \bar{\rho})A/r_0 + r_0}{1 - (\rho^2 + \bar{\rho}^2) + \rho^2\bar{\rho}^2}$$

Since  $\rho\bar{\rho} = b_2$  and  $\rho + \bar{\rho} = -b_1$  it follows that

$$\rho^2 + \bar{\rho}^2 = (\rho + \bar{\rho})^2 - 2\rho\bar{\rho} = b_1^2 - 2b_2$$

and thus,

$$\xi_0 = \frac{b_2r_0 - b_1A/r_0 + r_0}{1 - (b_1^2 - 2b_2) + b_2^2} = \frac{r_0^2(b_2 + 1) - Ab_1}{r_0[(b_2 + 1)^2 - b_1^2]}.$$

As expected,  $\xi_0 \in \mathbb{R}$ . A similar calculation yields

$$\xi_1 = \frac{A(b_2 + 1) - r_0^2b_1}{r_0[(b_2 + 1)^2 - b_1^2]}.$$

If  $0 < b_2 < 1$  then  $|\rho| = \mu = \sqrt{b_2} < 1$ . In this case, every solution of (24) converges to a 2-cycle  $\xi_0$  and  $\xi_1$ . These limit cycles depend on  $r_0$  and thus, on the initial values  $x_{-2}, x_{-1}, x_0$  in the sense that all initial points on the plane  $u + b_1v + b_2w = r_0$  converge to the same limit cycle. However, if  $b_2 \geq 1$  then other types of solutions, including unbounded solutions are possible for (24) that are driven by the homogeneous part of (27). To observe the 2-cycles numerically it is necessary to use the initial values

$$x_{-2} = \xi_0, \quad x_{-1} = \xi_1, \quad x_0 = r_0 - b_1\xi_1 - b_2\xi_0 = \xi_0.$$

Going in a different direction, if the function  $g$  in (28) has a 3-cycle then as is well-known, it has cycles of all possible lengths. In this case, if  $0 < b_2 < 1$  then (24) also has cycles of all possible lengths. A set of parameter values that imply this situation is  $A = 1$ ,  $C = 1 - a_0 - b_1$  and  $B = -\sqrt{3}$ ; see [2]. In this case, (26) is

$$r_{n+1} = \frac{1}{r_n} - \sqrt{3} + r_n$$

and its 3-cycle is found to be

$$\sigma_0 = \frac{2}{\sqrt{3}} \left(1 + \cos \frac{\pi}{9}\right), \quad \sigma_1 = g(\sigma_0), \quad \sigma_2 = g(\sigma_1).$$

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